Extrapolation of nucleus-nucleus cross-section to cosmic ray energies using geometrical model

Zbigniew Plebaniak

National Centre for Nuclear Research, Łódź, Poland

July 18, 2017
1. Description of proton-proton scattering in geometrical picture

2. Nucleus-Nucleus cross section
   - Proton-Nucleus case
   - Nucleus-Nucleus case

3. Conclusions
The differential elastic cross section is described by elastic scattering amplitude $F(s, t)$ as follows:

$$
\frac{d \sigma_{el}}{d|t|} = \pi |S(t)|^2
$$

(1)

The relation between interaction geometry and the momentum transfer space is defined with the Fourier transform.

$$
S(t) = \frac{i}{2\pi} \int e^{ibt} \left( 1 - e^{i\chi(b)} \right) d^2b
$$

(2)

where $b$ is the impact parameter.

The inverse transformation

$$
1 - e^{i\chi(b)} = \frac{1}{2\pi i} \int e^{-ibt} S(t) d^2t
$$

(3)

suggest the analogy of the proposed description of the collision to the transmission of the light through the gray medium.
Geometrical picture for p-p collisions

Eikonal $\chi(s, b)$ may be expressed by:

$$\chi(s, b) = (\lambda(s) + i)\Omega(b, s)$$  \hspace{1cm} (4)

Using this formalism, cross sections can be described as follows:

$$\sigma_{tot}(s) = 2 \int \left[ 1 - \Re(e^{i\chi(b, s)}) \right] d^2 b$$ \hspace{1cm} (5)

$$\sigma_{el}(s) = \int \left[ 1 - e^{i\chi(b, s)} \right]^2 d^2 b$$ \hspace{1cm} (6)

$$\sigma_{inel}(s) = \int 1 - \left[ e^{i\chi(b, s)} \right]^2 d^2 b$$ \hspace{1cm} (7)

Transmission coefficient $\Omega(b, s)$ was introduced by Chou and Yang in 1968, and can be described by following convolution:

$$\Omega(b) = \Omega(b) = iK_{pp} \int_{-\infty}^{\infty} \int D(b - b')D(b') \, d^2 b'$$ \hspace{1cm} (8)
Geometrical picture for p-p collisions

\[ \Omega(b) = \Omega(b) = iK_{pp} \int \int_{-\infty}^{\infty} D(b - b')D(b') \, d^2b' , \quad (9) \]

where \( D(b) \) is a hadron profile function represents the integral of density function \( \rho(x, y, z) \) in \( z \) direction as a collision axis:

\[ D(b) = \int_{-\infty}^{\infty} \rho(x, y, z) \, dz , \quad (10) \]

One of the simplest hadronic matter distributions \( \rho(x, y, z) \) we can imagine is the exponential decreasing of hadron matter density in radius:

\[ \rho_h(r) = \frac{m^3_h}{8\pi} e^{-m_h |r|} . \quad (11) \]
To obtain more satisfying description of elastic scattering, we introduced sum of two exponents with different slopes $m_1$ and $m_2$ and two different normalization factors $c_1$ and $c_2$ instead of one.

**Proposed hadron matter distribution**

$$\rho_h(r) = \frac{1}{8\pi} \left( c_1 m_1^3 e^{-m_1|r|} + c_2 m_2^3 e^{-m_2|r|} \right)$$  \hspace{1cm} (12)
Geometrical picture for p-p collisions

Results - fits to the differential elastic cross-section

Figure 3: The differential elastic cross sections from our model for c.m.s energies of 19 GeV, 546 GeV and 7 TeV shown as a function of the ($|t| \times \sigma_{tot}$)
Figure 4: Values of the elastic, inelastic and total cross section calculated with our model as a function of the interaction energy compared with the measurements. Solid line represents total cross section, dashed inelastic and dotted line elastic cross section predictions.

The optical model developed further by Czyż and co-workers led eventually to the wounded-nucleon picture to hadron- and nucleus-nucleus collisions. The formulas for nuclei cross sections exiting in the literature under the name of ”Glauber” (or ”optical”, ”eikonal”), sometimes called ”multiple diffraction”, ”multiple scattering”. They are not always the same, the chain of adopted approximations connects one with the others. Some of them could be questioned, so we would like to examined them briefly below.
The essence of the Glauber approximation is the natural assumption than resulting amplitude phase shift of the collision is the sum of all possible A individual nucleon-nucleon phase shifts

$$\chi_A(b, \{d\}) = \sum_{j=1}^{A} \chi(b - d_j)$$  \hspace{1cm} (13)

where $\{d\}$ is a set of nucleon positions in the nucleus ($d_j$ is a position of the $j$th nucleon in the plane perpendicular to the interaction axis). The scattering amplitude is

$$S(t) = \frac{i}{2\pi} \int e^{ibt} d^2b \int |\varphi(\{d\})|^2 \left\{ 1 - e^{i\chi_A(b,\{d\})} \right\} \prod_{j=1}^{A} d^2 d_j$$  \hspace{1cm} (14)

where the function $\varphi$ is the wave-function of the nucleus with nucleons distribution given by the $\{d\}$. 
This general formula is a subject of subsequent approximations leading to the set of consecutively simpler equation for the collision cross sections. First we set no space correlation between nucleons. With the universal nucleon distribution within the nucleus $\rho$ we obtain

$$|\varphi(\{d\})|^2 = \prod_{j=1}^{A} \rho_j(d_j)$$  \hspace{1cm} (15)$$

with the normalization ($\int \rho_j(r_j) d^3r_j = 1$). Next quite obvious approximation is that off individual sub-collisions are the same, having universal nucleon-nucleon phase-shifts dependence $\chi$. With this we have

$$S(t) = \frac{i}{2\pi} \int e^{ibt} d^2b \int \prod_{j=1}^{A} \rho_j(d_j) \left\{ 1 - e^{i\sum_{j=1}^{A} \chi(b-d_j)} \right\} d^2d_j =$$

$$= \frac{i}{2\pi} \int e^{ibt} d^2b \left\{ 1 - \int \prod_{j=1}^{A} \rho_j(d_j) e^{i\chi(b-d_j)} d^2d_j \right\}$$  \hspace{1cm} (16)$$
On the other hand, the scattering process can be treated as the single collision process with its own nuclear phase shift $\chi_{\text{opt}}(b)$

$$S(t) = \frac{i}{2\pi} \int e^{itb} \left\{ 1 - e^{i\chi_{\text{opt}}(b)} \right\} d^2b$$

(17)

The comparison of Eqs.(16) and (17) leads to the relation between the opacity for the nucleus and the single nucleon:

$$e^{i\chi_{\text{opt}}(b)} = \int |\varphi(\{d\})|^2 e^{i\sum_{j=1}^{A} \chi_j(b-d_j)} d^2d_j = \left\langle e^{i\chi(b,\{d\})} \right\rangle$$

(18)
Proton-Nucleus case

The calculation of the integral in Eq.(18) needs the knowledge of the fluctuation of the nucleus shape. For our averaging integral, we assume that the nucleus of a given number of constituting nucleons \(A\) is always the same. The shape of the 'average nucleus' we adopted from the Lund model in the form of Woods-Saxon distribution

\[
\rho_A(d) = \frac{\rho_0}{1 + \exp\left(\frac{d-d_0}{\delta}\right)}
\]

(19)

(for nuclei heavier than oxygen; the Gaussian for lighter nuclei) with two parameters \(d_0\) the 'half-density' radius and \(\delta\) related to the 'surface thickness' of the nucleus, respectively. For the convenience the density \(\varphi\) is normalized to \(A\).

If the number of nucleons \(A\) tends to infinity keeping obviously constant the value of the total nucleus opacity then

\[
\chi_{\text{opt}}(b) = i \int d^2d \, \rho_A(d) \left(1 - e^{i\chi(b-d)}\right)
\]

(20)
Proton-Nucleus case

Opacity of the nucleon is a sum of many (infinitely many, but actually $A > 6$ should be big enough (Glauber, 1957)) of independent nucleons of a small (infinitely small, as $A$ goes to infinity) scattering centers. Using Eq.(17) and the optical theorem the integration in Eq.(20) can be performed and for the case of pure imaginary scattering amplitude we obtain

$$\chi_{\text{opt}}(b) = \frac{1}{2} \sigma_{pp}^{\text{tot}} \rho_A(b) \quad (21)$$

The opacity given by Eq.(21) can be substituted to Eq.(17) and, e.g., the inelastic cross section could be eventually obtained

$$\sigma_{\text{inel}}^{pA} = \int \left[ 1 - e^{-\sigma_{pp}^{\text{tot}} \rho_A(b)} \right] d^2b \quad , \quad (22)$$

$$= \int \left\{ 1 - \left[ 1 - \sigma_{pp}^{\text{tot}} \frac{\rho_A(b)}{A} \right]^A \right\} d^2b \quad . \quad (23)$$

Both formulas, Eq.(22) and Eq.(23) define the so-called ”point nucleon” approximation.
Figure 5: Results of Glauber theory for p-Air collisions.
Figure 6: Comparison of $\sigma_{p-Air}^{inel}$ calculated with various method with experimental data. Pure Glauber theory - red line, ”point-nucleon” approximation - blue line.
Nucleus-Nucleus case

The treatment of nucleon-nucleus presented above can be extended to the case of nuclei collisions with the amplitude defined as

\[ S(t) = \frac{i}{2\pi} \int e^{itb} d^2b \quad (24) \]

\[ \left\{ 1 - \int d^2d_i \prod_{i=1}^{A} \rho_A(d_i) \int d^2d_j \prod_{j=1}^{B} \rho_B(d_j) e^{i\chi(b-d_i-d_j)} \right\} = \]

\[ = \frac{i}{2\pi} \int e^{itb} d^2b \left\{ 1 - \int d^2d_i \rho_{AB}(d_i) e^{i\chi(b-d_i)} \right\} \quad (25) \]

with

\[ \rho_{AB}(b) = \int d^2d_i \prod_{i=1}^{A} \rho_A(b - d_i) \int d^2d_j \prod_{j=1}^{B} \rho_B(d_j) \quad (26) \]
Using Eq.(17) we can define the overall nucleus-nucleus opacity

\[ e^{i\chi_{AB}(b)} = \int d^2d_i \rho_{AB}(\{d\}) e^{i \sum_{i=1}^{A} \chi(b-d_i)} = \langle e^{i\chi(b,\{d_{A}\},\{d_{B}\})} \rangle \]  

(27)

Thus the nucleus-nucleus scattering amplitude is, with the analogy to Eq.(2)

\[ S(t) = \frac{i}{2\pi} \int e^{itb} \left( 1 - e^{i\chi_{AB}(b)} \right) d^2b . \]  

(28)
The ”big nucleus” and ”point nucleon” approximation can be used also in this case leading to

\[
\sigma_{\text{inel}}^{AB} = \int \left[ 1 - e^{-\sigma_{\text{tot}}^{pp} \rho_{AB}(b)} \right] d^2b \quad , \quad (29)
\]

\[
= \int \left\{ 1 - \left[ 1 - \sigma_{\text{tot}}^{pp} \frac{\rho_{AB}(b)}{AB} \right]^A_B \right\} d^2b \quad . \quad (30)
\]
Figure 7: Comparison of $\sigma_{Fe-Air}^{inel}$ calculated with Glauber theory (red line) with "point-nucleon" approximation (blue line). PRELIMINARY !!!
Conclusions

- Presented model for p-p collisions works very well.
- Obtained hadron matter distribution profiles allows for calculations of p-N and N-N cross-section using Glauber theory.
- Calculations for heavier nuclei needs more attention.